16.2. Line integrals

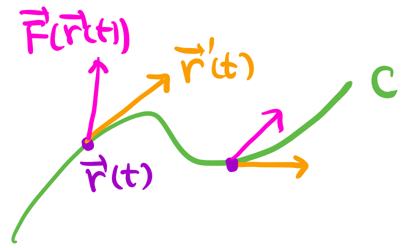
Def Let C be a curve parametrized by $\vec{r}(t)$ on $a \le t \le b$.

(1) Given a scalar function f, its <u>line integral</u> along the curve C is

$$\int_C f\,ds := \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

(2) Given a vector field $\vec{F}$, its <u>line integral</u> along the curve C is

$$\int_C \vec{F}\cdot d\vec{r} := \int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\text{vector}}\cdot \vec{r}\,'(t)\,dt$$

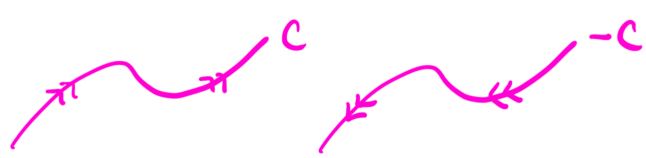


Note (1) If $\vec{F}$ is a force field, then $\int_C \vec{F}\cdot d\vec{r}$ is equal to the work done by $\vec{F}$ along C.

(2) The length of C is $\int_C 1\,ds = \int_a^b |\vec{r}\,'(t)|\,dt$

(3) The line integral of a vector field depends on the orientation of the curve :

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$



where $-C$ is the curve C with the opposite orientation.

cf. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(4) For a vector field $\vec{F} = (P, Q, R)$, we also write

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Ex Consider the helix C parametrized by

$\vec{r}(t) = (\cos t, \sin t, t)$ with $0 \le t \le 2\pi$.

Find its center of mass with density $\rho(x,y,z) = z$.

Sol $m = \int_C \rho(x,y,z)\,ds = \int_C z\,ds = \int_0^{2\pi} \underbrace{z(\vec{r}(t))}_{= t} |\vec{r}'(t)|\,dt.$

$\vec{r}'(t) = (-\sin t, \cos t, 1) \Rightarrow |\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$

$\Rightarrow m = \int_0^{2\pi} \sqrt{2}\,t\,dt = \frac{\sqrt{2}}{2} t^2 \Big|_{t=0}^{t=2\pi} = 2\sqrt{2}\pi^2.$

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y,z)\,ds = \frac{1}{2\sqrt{2}\pi^2}\int_C xz\,ds$$

$$= \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} x(\vec{r}(t))\,z(\vec{r}(t))\,|\vec{r}'(t)|\,dt$$

$$= \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} \sqrt{2}\,t\cos t\,dt \underset{\uparrow}{=} \frac{1}{2\pi^2}(t\sin t + \cos t)\Big|_{t=0}^{t=2\pi} = 0.$$

Integration by parts

$$\bar{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,ds = \frac{1}{2\sqrt{2}\pi^2}\int_C yz\,ds$$

$$= \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} y(\vec{r}(t))\,z(\vec{r}(t))\,|\vec{r}'(t)|\,dt$$

$$= \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} \sqrt{2}\,t\sin t\,dt \underset{\uparrow}{=} \frac{1}{2\pi^2}(\sin t - t\cos t)\Big|_{t=0}^{t=2\pi} = -\frac{1}{\pi}.$$

Integration by parts

$$\bar{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,ds = \frac{1}{2\sqrt{2}\pi^2}\int_C z^2\,ds$$

$$= \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} z(\vec{r}(t))^2\,|\vec{r}'(t)|\,dt = \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} \sqrt{2}\,t^2\,dt = \frac{t^3}{6\pi^2}\Big|_{t=0}^{t=2\pi} = \frac{4\pi}{3}$$

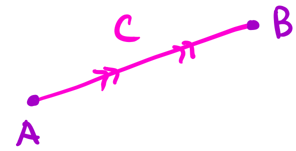
$\Rightarrow$ The center of mass is $\boxed{\left(0, -\frac{1}{\pi}, \frac{4\pi}{3}\right)}$

Ex Find the work done by the force field

$\vec{F}(x,y,z) = (x+y,\ y^2-z,\ 2z)$

along the line segment C from $(0,0,1)$ to $(2,1,0)$.

Sol Set $A=(0,0,1)$ and $B=(2,1,0)$



A direction vector is $\overrightarrow{AB} = (2,1,-1)$

$*$ $\overrightarrow{BA}$ gives the opposite orientation!

The line segment C is parametrized by

$\vec{r}(t) = (0+2t,\ 0+t,\ 1-t) = (2t,\ t,\ 1-t)$ on $0 \le t \le 1$.

$(\vec{r}(0) = (0,0,1) = A,\ \ \vec{r}(1) = (2,1,0) = B)$

The work done by $\vec{F}$ is

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt.$$

$\vec{F}(\vec{r}(t)) = (2t+t,\ t^2-(1-t),\ 2(1-t)) = (3t,\ t^2+t-1,\ 2t-2)$.

$\vec{r}\,'(t) = (2,1,-1)$

$\Rightarrow \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 2\cdot 3t + 1\cdot(t^2+t-1) - 1\cdot(2t-2) = t^2+9t-3$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^1 t^2+9t-3\,dt = \frac{t^3}{3}+\frac{9}{2}t^2-3t\Big|_{t=0}^{t=1} = \boxed{\frac{11}{6}}$$

Note Without the correct orientation, your answer would be incorrect.

Ex The vortex field $\vec{V}$ is defined by

$$\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

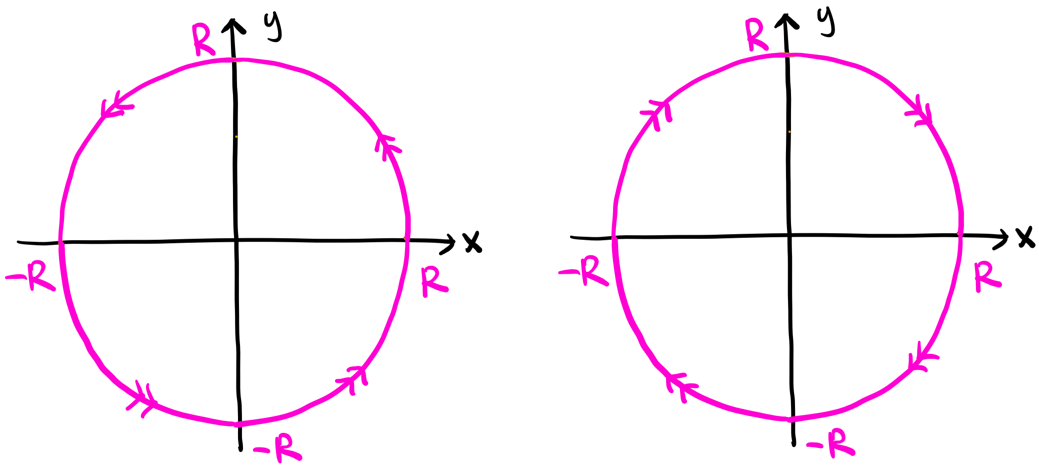
Find $\int_C \vec{V}\cdot d\vec{r}$ where C is a circle centered at $(0,0)$ with counterclockwise orientation.

Sol Let R be the radius of C.

Then C is parametrized by

$\vec{r}(t) = (R\cos t, R\sin t)$ with $0 \le t \le 2\pi$.

$*$ $\vec{s}(t) = (R\sin t, R\cos t)$ gives the opposite orientation.



$\vec{r}(t) = (R\cos t, R\sin t)$ $\quad\quad$ $\vec{s}(t) = (R\sin t, R\cos t)$

$$\int_C \vec{V}\cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\vec{V}(\vec{r}(t)) \underset{\substack{\uparrow \\ x^2+y^2=R^2 \text{ on } C}}{=} \left(-\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2}\right) = \left(-\frac{\sin t}{R}, \frac{\cos t}{R}\right)$$

$$\vec{r}\,'(t) = (-R\sin t, R\cos t)$$

$$\vec{V}(\vec{r}(t))\cdot\vec{r}\,'(t) = -\frac{\sin t}{R}\cdot(-R\sin t) + \frac{\cos t}{R}\cdot R\cos t = 1.$$

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$$ independent of the radius.